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A Note on the Derivation of the Schrödinger–Langevin Equation

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It is shown that the dynamics of open systems interacting with a chaotic environment can be formulated in a quantum mechanical scheme by means of Nelson's stochastic quantization procedures. As a result, a wave equation for a particle in the chaotic environment is found to be of the Schrödinger–Langevin type associated with an additional nonlinear and random potential.

KEY WORDS: Stochastic quantization; Schrödinger–Langevin equation; Quantum mechanics of open systems.

Some time ago, to describe the dynamical behavior of an open system interacting with chaotic environments, Chandrasekhar and others argued for the classical Langevin equation

$$m\ddot{Q}(t) = -\text{grad}\, V(Q(t), t) - \gamma(t)\dot{Q}(t) + A(t) \tag{1}$$

where $Q(t) = (Q^1(t),..., Q^n(t))$ denote reduced coordinate variables of the open system, $V(Q, t)$ a usual potential function, $\gamma(t)$ a friction coefficient, $A(t)$ a Gaussian white-noise random force, and m a mass parameter.[1]

Recently Kostin[2] has introduced heuristically the quantum mechanical version of Eq. (1) as

$$i\hbar \frac{\partial}{\partial t}\psi(Q, t) = \left[-\frac{\hbar^2}{2m}\text{div grad} + V(Q, t) - Q\cdot A(t) + \frac{i\hbar}{2m}\gamma(t)\log\frac{\bar{\psi}}{\psi} + W(t)\right]\psi(Q, t) \tag{2}$$

where $W(t) = -(i\hbar/2m)\gamma(t)\int |\psi|^2 \log(\bar{\psi}/\psi)\, d^nQ$ and ψ denotes a wave

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function or a probability amplitude of the open system; the probability of finding coordinate variables Q in the volume element d^nQ at time t is given by $|\psi(Q, t)|^2 \, d^nQ$. Equation (2) is called the Schrödinger–Langevin equation by Kostin and plays an important role in clarifying the quantum theoretical behavior of open systems. However, Kostin's derivation of Eq. (2) is simply heuristic.

The purpose of the present note is to derive the Schrödinger–Langevin equation (2) directly from the classical Langevin equation in the stochastic framework of quantum mechanics proposed by Nelson.[3]

Nelson's quantization method consists in utilizing fundamental notions of stochastic processes and so has been called "stochastic quantization" by Guerra.[4] Because of the absence of the physically meaningful Hamiltonian or Lagrangian for the open system with dissipations, the only available quantization method seems to be the stochastic one, which demands no Lagrangian or Hamiltonian, but Newton's equation of motion in the generalized sense.

The first basic assumption of the stochastic quantization is that *the coordinate variables $Q(t)$ are given by a diffusion process which satisfies the stochastic differential equation*

$$dQ(t) = b(Q(t), t)\, dt + B(dt) \tag{3}$$

where $b(Q, t)$ denotes a drift term to be determined and $B(t)$ stands for an n-dimensional standard Brownian motion with a diffusion coefficient $\hbar/2m$ ($h = 2\pi\hbar$ is Planck's constant). The probability density defined as $\rho(Q, t)\, d^nQ = \mathrm{Prob}\{Q(t) \in d^nQ\}$ satisfies the Fokker–Planck equation

$$(\partial/\partial t)\rho = -\mathrm{div}(b\rho) + (\hbar/2m)\, \mathrm{div}\, \mathrm{grad}\, \rho \tag{4}$$

Before we give our second assumption, we summarize the definitions of mean velocity v and mean acceleration a of the process $Q(t)$:

$$v(Q(t), t) \equiv \tfrac{1}{2}(D + D_*)Q(t) \tag{5}$$

$$a(Q(t), t) \equiv \tfrac{1}{2}(DD_* + D_*D)Q(t) \tag{6}$$

$$\begin{aligned} Df(Q(t), t) &\equiv \lim_{h \downarrow 0} \frac{1}{h} E[f(Q(t+h), t+h) - f(Q(t), t) | Q(t)] \\ &= \left(\frac{\partial}{\partial t} + b \cdot \mathrm{grad} + \frac{\hbar}{2m}\, \mathrm{div}\, \mathrm{grad}\right) f(Q(t), t) \end{aligned} \tag{7}$$

$$\begin{aligned} D_*f(Q(t), t) &\equiv \lim_{h \downarrow 0} \frac{1}{h} E[f(Q(t), t) - f(Q(t-h), t-h) | Q(t)] \\ &= \left(\frac{\partial}{\partial t} + b_* \cdot \mathrm{grad} - \frac{\hbar}{2m}\, \mathrm{div}\, \mathrm{grad}\right) f(Q(t), t) \end{aligned} \tag{8}$$

where $b_* = b - (\hbar/m)$ grad log ρ and $E[\cdot \,|\, Q(t)]$ is the conditional expectation with respect to $Q(t)$.

The second basic assumption is that *the Langevin equation (1) is expressed in terms of mean velocity and acceleration, i.e.,*

$$ma(Q(t), t) = -\text{grad}\, V(Q(t), t) - \gamma(t)v(Q(t), t) + A(t) \tag{9}$$

This can be rewritten by making use of Eqs. (5)–(8) as

$$m \frac{\partial}{\partial t} v + m(v \cdot \text{grad})v - \frac{\hbar^2}{2m} \text{grad} \frac{\text{div grad} \sqrt{\rho}}{\sqrt{\rho}} = -\text{grad}\, V - \gamma v + A \tag{10}$$

Finally, we assume that *the momentum is a gradient of a smooth function S as*

$$mv(Q(t), t) = \hbar \,\text{grad}\, S(Q(t), t) \tag{11}$$

Then, introducing the wave function or the probability amplitude

$$\psi \equiv \sqrt{\rho} \exp iS, \qquad i^2 = -1 \tag{12}$$

one obtains a Schrödinger-type equation with an additional nonlinear term and a random potential from Eqs. (4) and (10):

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(Q, t) &= \left[-\frac{\hbar^2}{2m} \text{div grad} + V(Q, t) - Q \cdot A(t) \right. \\ &\qquad \left. + \frac{\hbar}{m} \gamma(t) S(Q, t) + \alpha(t) \right] \psi(Q, t) \\ &= \left[-\frac{\hbar^2}{2m} \text{div grad} + V(Q, t) - Q \cdot A(t) \right. \\ &\qquad \left. + \frac{i\hbar}{2m} \gamma(t) \log \frac{\bar{\psi}}{\psi} + \alpha(t) \right] \psi(Q, t) \end{aligned} \tag{13}$$

Note that $\alpha(t)$ may be an arbitrary real function of time t and Eq. (13) coincides with the Schrödinger–Langevin equation (2) with the identification

$$\alpha(t) \equiv W(t) = -\frac{\hbar}{m} \gamma(t) \int |\psi|^2 S \, d^n Q = -\frac{i\hbar}{2m} \gamma(t) \int |\psi|^2 \log \frac{\bar{\psi}}{\psi} \, d^n Q \tag{14}$$

It may be worthwhile to point out that the probabilistic interpretation for the wave function evidently holds:

$$|\psi(Q, t)|^2 \, d^n Q = \rho(Q, t) \, d^n Q \tag{15}$$

even though Eq. (13) is no longer linear.

NOTE ADDED IN PROOF

After the completion of this work, we were informed that Dr. B. K. Skagerstam [*Phys. Lett.* **58B**:21 (1975)] had mentioned the possibility of a derivation of the Schrödinger–Langevin equation using Nelson's stochastic quantization.

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